International Journal of Theoretical Physics, Vol. 4, No. 5 (1971), pp. 327–338

A Problem of Information Gain by Quantal Measurements

H. J. GROENEWOLD

Institute for Theoretical Physics, State University, Groningen, Netherlands

Received: 17 *March* 1971

Abstract

For the average information gain by a quantal measurement of the first kind, an expression is suggested for a lower bound, which turns out to be non-negative.

1. *Introduction*

We envisage a statistical ensemble of a very large number of mutually independent identical quantal object systems. All samples are constructed in the same way. They have a small number of degrees of freedom, so that their observables may be represented on a corresponding Hilbert space $\mathscr{H}$. Moreover, all samples are prepared and selected in the same way, so that the state of the ensemble may be represented by a Hermitean positive statistical operator $\mathbf{k}$ on $\mathscr{H}$. Suppose $\mathbf{k}$ to be normalised by

$$\operatorname{tr}\mathbf{k} = 1 \tag{1.1}$$

Now we perform on all individual samples a quantal measurement of the first kind (Pauli, 1933). This is an abstract, highly schematised kind of measurement, which distinguishes between certain orthogonal subspaces $\mathscr{H}_m$ of $\mathscr{H}$. For simplicity we shall only consider the case when m has a discrete spectrum. If the spectrum is partially or entirely continuous, sums over m in formal expressions have in general to be replaced partially or entirely by integrals. Their interpretation has then to be adapted to the situation that for continuous m neither the measurement nor the reading can strictly be maximal. But in our considerations we shall use, in particular, a geometrical Euclidean representation, for which a generalisation from a discrete to a continuous spectrum is by no means obvious.

We make the system of subspaces $\mathscr{H}_m$ complete by adding the complementary subspace, which represents the incident of no definitive measuring result. The orthogonality and completeness of the $\mathscr{H}_m$'s and their dimension

d_m are then in terms of their Hermitean projectors $\mathbf{P}_m$ expressed by

$$\mathbf{P}_m\mathbf{P}_n = \delta_{mn}\mathbf{P}_m; \qquad \sum_m \mathbf{P}_m = \mathbf{1}; \qquad \mathrm{tr}\,\mathbf{P}_m = d_m \tag{1.2}$$

If all $d_m = 1$, the measurement is called maximal. If there is just one $\mathbf{P}_m = \mathbf{1}$, it is called minimal or trivial; in fact it is then no measurement at all.

Imagine that for every sample the measurement (the intermediate coupling between every object system and the corresponding measuring instrument) and its result are entirely automatically performed and recorded (first step). Afterwards the records, which are stored in some classical macro-code, are read by an observer (second step). The ensemble, which before the measurement was represented by the original statistical operator

$$\mathbf{k} = \sum_{mn} \mathbf{P}_m\mathbf{k}\mathbf{P}_n \tag{1.3}$$

is after the first step represented by (Lüders, 1951)

$$\mathbf{k}' = \sum_m \mathbf{P}_m\mathbf{k}\mathbf{P}_m = \sum_m (\mathrm{tr}\,\mathbf{P}_m\mathbf{k})(\mathbf{P}_m\mathbf{k}\mathbf{P}_m/\mathrm{tr}\,\mathbf{P}_m\mathbf{k}) \tag{1.4}$$

which I shall refer to as the truncated statistical operator. We may say that, owing to the intermediate interaction between the object system and the corresponding measuring instrument during the first step, the correlations between different subspaces $\mathscr{H}_m$ and $\mathscr{H}_n$ are completely lost, whereas the correlations inside every subspace $\mathscr{H}_m$ are completely conserved.

The degree of mixing of the ensemble before and after the first step may (in suitable units) be measured by the entropy expressions

$$S = -\mathrm{tr}\,\mathbf{k}\ln\mathbf{k} \tag{1.5}$$

$$S' = -\mathrm{tr}\,\mathbf{k}'\ln\mathbf{k}' \tag{1.6}$$

If $S = 0$, the original state is indivisible or pure. Because the transformation from $\mathbf{k}$ to $\mathbf{k}'$ is dissipative (Groenewold, 1964), the entropy increase is non-negative

$$S' - S \geqslant 0 \tag{1.7}$$

It measures the loss of correlation between the subspaces or, in other words, the loss of information about the ensemble in the first step.

On the other hand new correlations are in the first step established between the object samples and the corresponding record samples in the composite ensemble of objects, measuring instruments and records. This correlation is fully exploited in the second step, in which all object samples corresponding to the same reading result 'm' of the records are collected into a subensemble represented by the normalised statistical operator

$$\mathbf{k}_m'' = \mathbf{P}_m\mathbf{k}\mathbf{P}_m/\mathrm{tr}\,\mathbf{P}_m\mathbf{k}; \qquad \mathrm{tr}\,\mathbf{k}_m'' = 1 \tag{1.8}$$

The non-negative normalised statistical frequencies or weights

$$w_m = \mathrm{tr}\,\mathbf{P}_m\mathbf{k}; \qquad w_m \geqslant 0; \qquad \sum_m W_m = 1 \tag{1.9}$$

of these selected subensembles may be interpreted as the probabilities of the recordings 'm'. The degree of mixing of such a subensemble is

$$S_m'' = -\mathrm{tr}\,\mathbf{k}_m''\ln\mathbf{k}_m'' \tag{1.10}$$

Its average over the entire ensemble is

$$S'' = \sum_m w_m S_m'' \tag{1.11}$$

In the second step the average decrease of the degree of mixing is (as it should be) equal to the non-negative average information from the classical macro-readings of all recorded samples

$$\begin{aligned} S' - S'' &= -\mathrm{tr}\left(\sum_m w_m\mathbf{k}_m''\right)\ln\left(\sum_m w_m\mathbf{k}_m''\right) + \sum_m w_m\,\mathrm{tr}\,\mathbf{k}_m''\ln\mathbf{k}_m'' \\ &= -\sum_m w_m\ln w_m \geqslant 0 \end{aligned} \tag{1.12}$$

So there is a non-negative loss (1.7) of information in the first step (which is characteristic for the quantal measurement) and a non-negative gain (1.12) of average information in the second step (which is characteristic for the classical reading). The problem of the present paper is, whether for an abstract ideal quantal measurement of the first kind with maximal reading, the overall information gain over the two steps, averaged over the entire ensemble, is always non-negative. In other words, whether

$$S - S'' \geqslant 0 \tag{1.13}$$

may either be verified by a rigorous proof or falsified by a counter example.

2. *Matrix Representation*

If we choose in every subspace $\mathscr{H}_m$ an orthonormal complete set of base vectors, they form together an orthonormal complete set in $\mathscr{H}$. The matrix of $\mathbf{k}$ with respect to this base may be divided into blocks with respect to the subspaces. In the first step from the original $\mathbf{k}$ to the truncated $\mathbf{k}'$ all non-diagonal blocks are reduced to zero, whereas the diagonal blocks remain unaffected. By a suitable unitary transformation the latter may be brought into pure diagonal form. In this representation the matrix elements of $\mathbf{k}'$ may be written

$$\delta_{mn}\delta_{rs}a_{mr}; \qquad a_{mr} \geqslant 0; \qquad \sum_r a_{mr} = w_m; \qquad \sum_m w_m = 1 \qquad (r, s = 1, 2, \ldots) \tag{2.1}$$

In the same representation the matrix elements of **k** in the diagonal blocks are also given by (2.1), whereas those in the non-diagonal blocks may be written in the form

$$\xi_{mr|ns}\sqrt{(a_{mr}a_{ns})};\qquad \xi_{mr|ns}=x_{mr|ns}\exp(i\alpha_{mr|ns})$$
$$=x_{ns|mr}\exp(-i\alpha_{ns|mr})=\xi^{*}_{ns|mr}\qquad (m\neq n)\quad (2.2)$$

The latter form is obvious for rows and columns with non-zero diagonal elements. For the remaining cases it is a consequence of the lemma, that wherever in a Hermitean positive matrix **k** a zero diagonal element occurs, all non-diagonal elements in the corresponding row and column must also be zero.

In order to prove this lemma†, consider a Hermitean square root $\mathbf{k}^{1/2}$ of **k** and two (column) vectors ψ and φ. Then owing to Hermitecity and the Schwarz inequality, we have

$$|(\psi,\mathbf{k}\varphi)|=|(\mathbf{k}^{1/2}\psi,\mathbf{k}^{1/2}\varphi)|\leqslant\|\mathbf{k}^{1/2}\psi\|\|\mathbf{k}^{1/2}\varphi\|$$
$$=\{(\psi,\mathbf{k}\psi)(\varphi,\mathbf{k}\varphi)\}^{1/2}\qquad (2.3)$$

Choosing for ψ and φ base column vectors each with one element equal to 1 and all other elements equal to 0, this leads for the matrix elements of **k** to the inequality

$$|k_{mr|ns}|\leqslant\{k_{mr|mr}k_{ns|ns}\}^{1/2}\qquad (2.4)$$

from which the lemma follows at once.

The effect of the first step in the measurement is now represented by the reduction of the absolute values $x_{mr|ns}$ of all parameters $\xi_{mr|ns}$ $(m\neq n)$ to zero. Those corresponding to zero diagonal elements of $\mathbf{k}'$ may from the beginning be chosen equal to zero.

The eigenvalues κ_j of the original normalised positive statistical operator **k** satisfy the determinantal equation

$$\det|\mathbf{k}-\kappa_j\mathbf{1}|=0\qquad (j=1,2,\ldots)\qquad (2.5)$$

and the conditions

$$\kappa_j\geqslant 0;\qquad \sum_j\kappa_j=1\qquad (2.6)$$

Owing to the latter conditions, **k** may be represented by a point with barycentric coordinates κ_j $(j=1,2,\ldots)$ in a regular Euclidean simplex of unit height. In case the Hilbert space $\mathscr{H}$ has a finite number N of dimensions, we need a $(N-1)$-dimensional simplex. Properly **k** is represented by the whole symmetric set $\{\kappa_1,\kappa_2,\ldots\}$ of all points obtained by permutation of the κ_j in $\{\kappa_1,\kappa_2,\ldots\}$. So actually we need only a part (for finite N a $(1/N!)$-th part) of the simplex.

† I owe this direct proof to H. J. Brascamp.

The degrees of mixing (1.5); (1.6) now read

$$S = -\sum_j \kappa_j \ln \kappa_j \tag{2.7}$$

$$S' = -\sum_{mr} a_{mr} \ln a_{mr} \tag{2.8}$$

The function (2.7) may be represented in an additional Euclidean dimension by a convex hypersurface over the simplex. I shall refer to it as the surface of the S-hill. It might also be represented without an additional dimension by the convex adiabatic surfaces of constant S in the simplex. If I speak about inward or outward in the simplex it will be meant with respect to the latter hypersurfaces. That corresponds to upward or downward respectively on the S-hill.

Now let us consider a fixed arbitrary truncated statistical operator $\mathbf{k}'$ represented by the symmetric set $\{a_{mr}\}$. All compatible original statistical operators $\mathbf{k}$, i.e. all $\mathbf{k}$ which in the first step would be transformed into this fixed $\mathbf{k}'$, may be found by variation of all parameters $\xi_{mr|ns}$ in (2.2) with due observance of the conditions (2.6) for non-negative $\mathbf{k}$. The corresponding representative sets $\{\kappa_j\}$ determine a region in the simplex, which in this sense is compatible with the truncated set $\{a_{mr}\}$. They also determine a compatible region on the surface of the S-hill. The highest points in the latter region correspond to the set $\{a_{mr}\}$ with an S-value given by (2.8). If we could find the lowest value $S_{\min}$ in this region and show that

$$S_{\min} - S'' \geq 0 \tag{2.9}$$

then (1.13) would be verified.

In as far as the diagonal representation of $\mathbf{k}'$ has zero diagonal elements, the corresponding rows and columns of all compatible $\mathbf{k}$ also contain in this representation zero elements only. The corresponding dimensions may simply be omitted from the Hilbert space $\mathscr{H}$ and from the Euclidean simplex. If that has been done, the representative set $\{a_{mr}\}$ of $\mathbf{k}'$ has only left positive barycentric coordinates and lies inside the simplex.

3. *Conjectured Lower Bound*

The convexity of the S-surface suggests that in order to descend considerably in the compatible region, it may help to reduce the number of non-zero eigenvalues κ_j and to make them as unequal as possible.

We start with rearranging the rows and columns of the $\mathbf{k}$-matrix in such a way, that diagonal elements a_{mr} with equal index r instead of equal index m are brought together in diagonal blocks. This rearrangement depends on the numbering of the diagonal elements a_{mr} in the original blocks. Therefore we consider separately each of all permutations of the a_{mr} for every fixed m.

For a given permutation we choose all parameters $\xi_{mr|ns}$ in the non-diagonal blocks ($r \neq s$) equal to zero. If in the diagonal blocks we choose

$$\xi_{mr|nr} = \exp(i\alpha_{mr} - i\alpha_{nr}) \qquad (m \neq n) \tag{3.1}$$

they may by a suitable unitary transformation be brought into diagonal form with one diagonal element equal to

$$v_r = \sum_m a_{mr} \qquad \left(v_r \geqslant 0; \sum_r v_r = 1\right) \tag{3.2}$$

and all other diagonal elements equal to zero. The degree of mixture of the corresponding compatible operator $\mathbf{k}_c$ is

$$S_c = -\sum_r v_r \ln v_r \tag{3.3}$$

Intermediate between any fixed $\mathbf{k}_c$ and $\mathbf{k}'$ we might define other special compatible operators $\mathbf{k}_d$ by dividing the new diagonal blocks (for fixed r) into smaller sub-blocks and choosing in the non-diagonal sub-blocks the $\xi_{mr|nr}$ equal to zero instead of (3.1). This may be done for all combinations of the a_{mr} for fixed r into sub-sets. $\mathbf{k}'$ is the extreme case of a $\mathbf{k}_d$ for which all sub-blocks are 1-dimensional. For every $\mathbf{k}_d$ which in this sense is intermediate between a certain $\mathbf{k}_c$ and $\mathbf{k}'$ the number of zero eigenvalues is also intermediate. Further (because the transformations from $\mathbf{k}_c$ to $\mathbf{k}_d$ and from $\mathbf{k}_d$ to $\mathbf{k}'$ are dissipative) we have then $S_c \leqslant S_d \leqslant S'$.

If we omit the dimensions of zero eigenvalues of $\mathbf{k}'$, its representative set lies inside the simplex. All $\mathbf{k}_d$ and $\mathbf{k}_c$ have still other zero eigenvalues and their representative sets lie on the boundary of the simplex.

Before looking for the lowest value of S_c, we check for all permutations that

$$S_c - S'' \geqslant 0 \tag{3.4}$$

or with (1.10); (1.11) and (3.3) that

$$-\sum_r v_r \ln v_r - \sum_m w_m \ln w_m + \sum_{mr} a_{mr} \ln a_{mr} \geqslant 0 \tag{3.5}$$

For this purpose we determine the minimum of the left-hand member of (3.5) under the subsidiary conditions

$$w_m = \sum_r a_{mr}; \qquad v_r = \sum_m a_{mr}; \qquad \sum_{mr} a_{mr} = 1; \qquad a_{mr} \geqslant 0 \tag{3.6}$$

The extremum of

$$-\sum_r \left(\sum_m a_{mr}\right) \ln \left(\sum_m a_{mr}\right) - \sum_m \left(\sum_r a_{mr}\right) \ln \left(\sum_r a_{mr}\right) + \sum_{mr} a_{mr} \ln a_{mr} + \lambda \left(\sum_{mr} a_{mr} - 1\right) \tag{3.7}$$

with Lagrangian multiplier λ is reached for

$$-1-\ln\left(\sum_n a_{nr}\right)-1-\ln\left(\sum_s a_{ms}\right)+1+\ln a_{mr}+\lambda=0 \tag{3.8}$$

for all m and r, i.e. for

$$a_{mr}=w_m v_r; \qquad \lambda=1 \tag{3.9}$$

The corresponding extremum of the left-hand member of (3.5) with (3.6) is zero and this is indeed a minimum.

Owing to the convexity of the S-function (2.7) the permutations of the a_{mr} for every fixed m which give the lowest value S_0 for S_c are obtained by numbering them in a non-increasing order

$$a_{mr}\geqslant a_{ms} \qquad \text{for } r<s \tag{3.10}$$

The eigenvalues r_r of the corresponding well-ordered compatible operator $\mathbf{k}_0$ are so to say more unequal than those of any other operator $\mathbf{k}_c$.

If we could show that S_0 is equal to the lowest S-value S_{min} in the compatible region of the S-surface

$$S_0=S_{min} \tag{3.11}$$

then (2.9) and hence (1.13) would be verified. I conjecture that (3.11) holds (even for continuous spectra, with sums replaced by integrals). In the following sections I shall give some comments on this conjecture.

4. *Compatible Regions*

In case of a maximal measurement ($d_m=1$ for all m) and also in the case of a pure original ensemble ($\mathbf{k}^2=\mathbf{k}$; $S=0$), all selected subensembles are pure ($\mathbf{k}_m''^2=\mathbf{k}_m''$; $S_m''=0$). In all these cases $\bar{S}''=0$ and then (1.13) is trivial. The case of a minimal measurement (only one $\mathscr{H}_m=\mathscr{H}$) is trivial in any case.

As the simplest non-trivial examples we consider a truncated $\mathbf{k}'$ with only a finite number N of non-zero eigenvalues. In order that the corresponding simplex may easily be drawn, we restrict N to 3 or 4 only. For $N=3$ the only non-trivial case is $d_1=1$; $d_2=2$. For $N=4$ there are three different non-trivial cases: (i) $d_1=1$; $d_2=3$, (ii) $d_1=d_2=2$ and (iii) $d_1=d_2=1$; $d_3=2$ (of course m may be numbered in various ways).

4.1. $N=3$

We write the matrix representation of a compatible $\mathbf{k}$ as

$$\mathbf{k}=\left(\begin{array}{c|cc} a & v\sqrt{(ab)} & \zeta\sqrt{(ac)} \\ \hline v^*\sqrt{(ab)} & b & 0 \\ \zeta^*\sqrt{(ac)} & 0 & c \end{array}\right) \tag{4.1.1}$$

with

$$a, b, c > 0; \qquad a + b + c = 1 \tag{4.1.2}$$

The eigenvalue equation then reads

$$-\kappa^3 + \kappa^2 - \{(1 - y^2)ab + (1 - z^2)ac + bc\}\kappa + \{1 - y^2 - z^2\}abc = 0 \tag{4.1.3}$$

independent of the phases of v and ζ. In order to satisfy additionally the first condition of (2.6), the expressions within the curled brackets have to be non-negative.

The simplex is a regular triangle. It appears that the compatible region is bounded by straight line segments parallel to the sides of the triangle. The

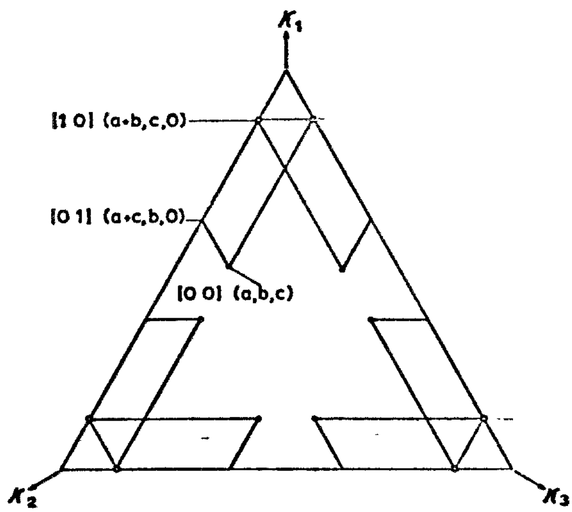

Figure 1.

typical shape depends on the relative magnitude of a, b, c and certain composite expressions. Figure 1 has been drawn for one choice with $a > b > c$.

For the admitted combinations of the values 0 and 1 for y^2 and z^2, the solutions of (4.1.3) represent the operators $\mathbf{k}_c$ defined in Section 3. The values of $[y^2 z^2]$ and of one of the corresponding permutations of $(\kappa_1, \kappa_2, \kappa_3)$ are indicated in Fig. 1. The representative symmetric sets form the corners of the compatible region $\mathbf{k}'$ and $\mathbf{k}_0$ in particular are represented by the most inward and outward set respectively. The latter forms the (six) corners of a truncated triangle which entirely encloses the compatible region. Therefore they determine the symmetric set of most outward points of the compatible region or lowest compatible points on the S-hill.

4.2. $N=4$

In all cases we have

$$a,b,c,d>0; \qquad a+b+c+d=1 \tag{4.2.1}$$

In case (i) we write

$$\mathbf{k}=\left(\begin{array}{c|ccc} a & \xi\sqrt{(ab)} & \upsilon\sqrt{(ac)} & \zeta\sqrt{(ad)} \\ \hline \xi^*\sqrt{(ab)} & b & 0 & 0 \\ \upsilon^*\sqrt{(ac)} & 0 & c & 0 \\ \zeta^*\sqrt{(ad)} & 0 & 0 & d \end{array}\right) \tag{4.2.2}$$

The corresponding eigenvalue equation reads

$$\begin{aligned} &\kappa^4-\kappa^3+\{(1-x^2)ab+(1-y^2)ac+(1-z^2)ad+bc+cd+db\}\kappa^2 \\ &\quad -\{(1-x^2-y^2)abc+(1-y^2-z^2)acd+(1-z^2-x^2)adb+bcd\}\kappa \\ &\quad +\{1-x^2-y^2-z^2\}abcd=0 \end{aligned} \tag{4.2.3}$$

independent of the phases of ξ, υ and ζ. In case (ii) we write

$$\mathbf{k}=\left(\begin{array}{cc|cc} a & 0 & \upsilon\sqrt{(ac)} & \zeta\sqrt{(ad)} \\ 0 & b & \rho\sqrt{(bc)} & \sigma\sqrt{(bd)} \\ \hline \upsilon^*\sqrt{(ac)} & \rho^*\sqrt{(bc)} & c & 0 \\ \zeta^*\sqrt{(ad)} & \sigma^*\sqrt{(bd)} & 0 & d \end{array}\right) \tag{4.2.4}$$

The eigenvalue equation now reads

$$\begin{aligned} &\kappa^4-\kappa^3+\{ab+(1-y^2)ac+(1-z^2)ad+(1-r^2)bc+(1-s^2)bd+cd\}\kappa^2 \\ &\quad -\{(1-y^2-r^2)abc+(1-z^2-s^2)abd+(1-y^2-z^2)acd+(1-r^2 \\ &\quad -s^2)bcd\}\kappa+\{1-y^2-z^2-r^2-s^2+y^2s^2+z^2r^2-2yzrs\cos\varphi\}abcd \\ &\qquad =0 \end{aligned} \tag{4.2.5}$$

where φ is the phase of $\upsilon\zeta^*\rho^*\sigma$. In case (iii) finally we write

$$\mathbf{k}=\left(\begin{array}{c|c|cc} a & \xi\sqrt{(ab)} & \upsilon\sqrt{(ac)} & \zeta\sqrt{(ad)} \\ \hline \xi^*\sqrt{(ab)} & b & \rho\sqrt{(bc)} & \sigma\sqrt{(bd)} \\ \hline \upsilon^*\sqrt{(ac)} & \rho^*\sqrt{(bc)} & c & 0 \\ \zeta^*\sqrt{(ad)} & \sigma^*\sqrt{(bd)} & 0 & d \end{array}\right) \tag{4.2.6}$$

The eigenvalue equation then reads

$$\begin{aligned} &\kappa^4-\kappa^3+\{(1-x^2)ab+(1-y^2)ac+(1-z^2)ad+(1-r^2)bc \\ &\quad +(1-s^2)bd+cd\}\kappa^2-\{(1-x^2-y^2-r^2)abc+(1-x^2-z^2-s^2)abd \\ &\quad +(1-y^2-z^2)acd+(1-r^2-s^2)bcd\}\kappa+\{1-x^2-y^2-z^2-r^2 \\ &\quad -s^2+y^2s^2+z^2r^2-2yzrs\cos\varphi+2xyr\cos\psi+2xzs\cos\chi\}abcd=0 \end{aligned} \tag{4.2.7}$$

where φ, ψ and χ are the phases of $\upsilon\zeta^*\rho^*\sigma$, $\xi\upsilon^*\rho$ and $\xi\zeta^*\sigma$ respectively.

The simplex is a regular tetrahedron. The compatible region appears to be bounded by flat surface segments parallel to the sides of the tetrahedron. The typical shape depends on the relative magnitude of a, b, c, d and certain composite expressions. It may consist of parts with crossing over connections along edges and even multiple connections around holes. Figures 2, 3 and 4 have tentatively been sketched for case (i), (ii) and (iii) respectively for one choice with $a > b > c > d$.

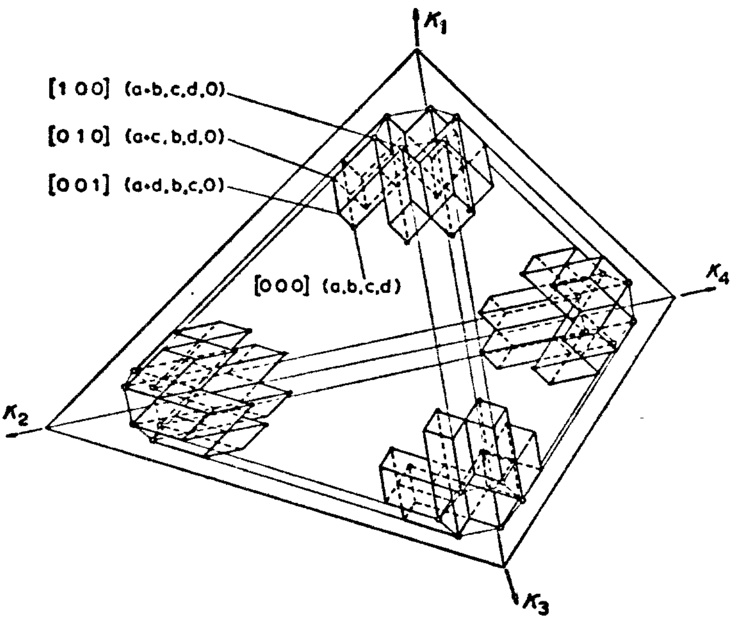

Figure 2.

For the admitted combinations of the values 0 and 1 for the relevant subset of x^2, y^2, z^2, r^2, s^2, the solutions of the eigenvalue equation in question represent the operators $\mathbf{k}_d$ and $\mathbf{k}_c$. At least some of the representative symmetric sets form the principal corners of the compatible region. The values of the relevant subset of

$$\begin{bmatrix} x^2 & y^2 & z^2 \\ & r^2 & s^2 \end{bmatrix}$$

and of one of the corresponding permutations of $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ are indicated in Figs. 2, 3 and 4 $\mathbf{k}'$ and $\mathbf{k}_0$ in particular are represented by the most inward and outward corners respectively. The latter forms the corners [24 in case (i); 12 in case (ii) or (iii)] of a truncated tetrahedron [cut off around the corners and in case (i) also around the edges], which entirely encloses the compatible region. Therefore they determine the set of most outward points of the compatible region or lowest compatible points on the S-hill.

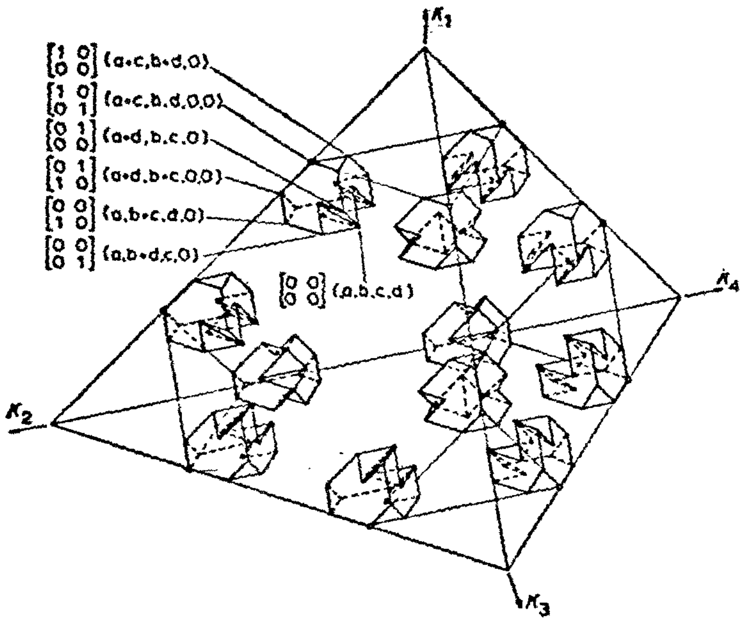

Figure 3.

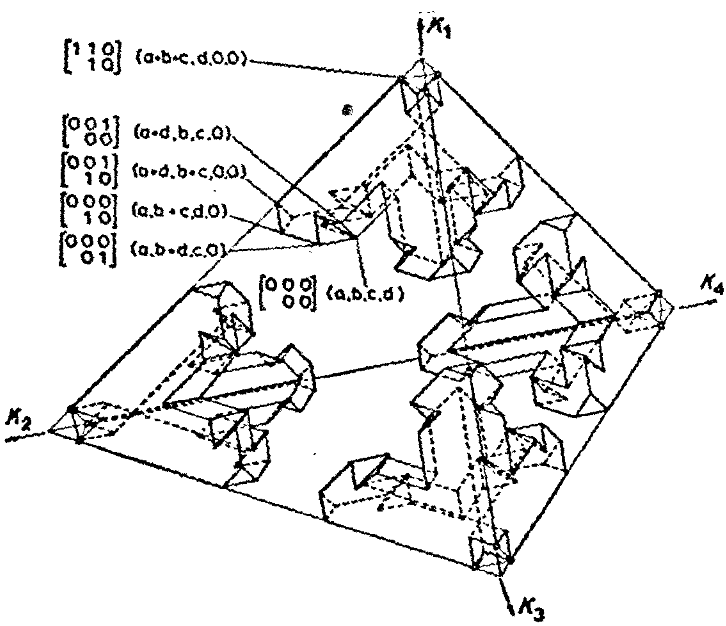

Figure 4.

4.3. $N > 4$

Also for a number $N > 4$ of non-zero eigenvalues of $\mathbf{k}'$, the representative set of $\mathbf{k}_0$ forms the complete set of corners (and at the same time most outward points) of a symmetric truncated simplex. Equation (3.11) would be proved if we could show that the compatible region could not reach outside this truncated simplex.

After inspection of the cases $N \leqslant 4$ it is perhaps not far-fetched to conjecture that in general the compatible region will be bounded by flat hypersurfaces parallel to the sides of the simplex, that the absolute values of the parameters $\xi_{mr\,ns}$ are bounded by

$$x_{mr|ns} \leqslant 1 \tag{4.3.1}$$

(apart from further mutual conditions), and that the whole compatible region does not reach outside the truncated simplex. It is in particular the last conjecture that counts.

If a proof of (3.11) might be given for finite N, generalisation to an infinite number would still require special care. Still more so would generalisation to partially or entirely continuous spectra of m. For such generalisation an entirely different kind of proof might be preferable.

5. *Conclusion*

All information has to be paid for somehow by a relatively high tax of negentropy. Nevertheless this tax may sometimes appear small in practical units, owing to the smallness of Boltzmann's constant k. In speaking about loss and gain of information, I leave in this paper such a negentropy tax expressly out of account.

It is peculiar that for quantal measurements, that information about the ensemble of object systems may be partially lost in the first step of coupling with the measuring instrument, and gained only in the second step of reading of the recorded measuring results. It is easy to design quantal measurements for which the loss is larger than the gain (e.g. by preventing maximal reading). But it appears difficult to see whether, for example the conjecture that in a general quantal measurement of the first kind with maximal reading the loss can never be larger than the gain, is correct.

The problem is rather academic. Only a relatively simple proof might be expected to contribute considerably to insight in quantum mechanics. It would not seem wise to spend much labour on finding a very complicated proof. The problem might be settled, if perhaps some day somebody would happen to run up against a counter example.

References

Groenewold, H. J. (1964). *Nuclear Physics*, **57B**, 112.
Lüders, G. (1951). *Annalen der Physik*, **8**, 322.
Pauli, W. (1933). *Handbuch der Physik*, **XXIV/1**, 152.